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Scattering of vector bosons by an asymmetric Hulthen potential

Kenan Sogut and Ali Havare

Mersin University, Department of Physics, Mersin 33143, Turkey

E-mail: kenansogut@gmail.com and ahavare@gmail.com

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Abstract
Exact solutions for the bound and scattering states of the spatially one-dimensional Duffin–Kemmer–Petiau equation for an asymmetric Hulthen potential are obtained in terms of the hypergeometric functions. These solutions are used to derive conditions for the existence of the transmission resonances. From this investigation, we show how the transmission coefficient depends on the energy of the particle and the strength of the potential barrier.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

Finding the solutions of the Schrödinger equation for various external potentials and discussing the bound and scattering states of the system is a fundamental problem of the non-relativistic quantum mechanics [1]. The low-energy limits of the bound and scattering states introduce interesting results; the potential supports a half-bound state (or zero-energy resonance) [2–4]. Reflection and transmission coefficients are calculated to be 1 and 0 in this limit, respectively. In non-relativistic scattering, if the external potential supports a bound state for the low momentum limit, then a half-bound state occurs and transmission resonances appear. In that case, the reflection coefficient vanishes and the transmission coefficient becomes unity. Bohm called such cases ‘transmission resonances’ and recognized that the condition for the existence of the transmission resonance was the same as the condition for a bound state energy eigenvalue [5].

Recently, Kennedy *et al* have generalized these phenomena to the relativistic case and showed that the transmission resonances appearing for Dirac particles scattered by a potential barrier have been related to the bound states for the corresponding potential well in the zero-momentum limit [6–8]. Thereby, their pioneering work on determining the transmission resonances in the low-momentum limit for a potential barrier in the Dirac equation has been extended to the spin-0 particles. The Klein–Gordon (for relativistic spin-0 particles) and Dirac (for relativistic spin-1/2 particles) equations are the most studied ones. These equations have been solved exactly for some symmetric and asymmetric potentials and the existence of the transmission resonances and supercritical states for these potentials have been discussed [9–14]. Among the studied symmetric potentials, Kennedy has solved the Dirac equation for Wood–Saxon potential barrier [8] and examined the transmission resonances in the low-momentum limit. He has shown that this potential barrier supports a half bound state at $E = -m$. After this study, Villalba *et al* have handled the transmission resonance phenomena in detail and investigated the transmission resonances for both scalar bosons and fermions for some external potentials [9–12].

Another important relativistic equation is the Duffin–Kemmer–Petiau (DKP), which describes both the massive scalar and vector bosons. Its covariant form resembles the Dirac equation with matrices satisfying the DKP algebra. Recently, a great deal of interest has been devoted to the DKP equation because of its applicability to the particle and nuclear physics, such as the study of the meson–nuclear interaction and deuteron-nucleus scattering. It has been studied in the context of the QCD and in the scattering of nucleus with exact solutions for some external potentials [15–19].

Unlike the Klein–Gordon and Dirac equations, the DKP equation has not been examined in the view of the existence of the transmission resonance phenomenon. In this paper, we are interested in solving the spatially one-dimensional DKP equation for the asymmetric form of the Hulthen potential [20] and deriving the condition for the existence of the transmission resonance. The Hulthen potential is extensively used in many parts of physics, such as nuclear and particle physics, solid state physics and chemical physics [21]. It involves many class of potentials used in physics, such as Coulomb, Cusp and Wood–Saxon potentials, with particular approaches of the parameters defined within. It reduces to the usual Hulten potential for $a = b$ and $q = \widetilde{q} = 1$, to the symmetric and asymmetric Cusp potential for $q = \widetilde{q} = 0$ and the Wood–Saxon potential for $a = b$ and $q = \widetilde{q} = -1$. The general form of the one-dimensional asymmetric Hulthen potential barrier is defined by

$$eA_0 = V(x) = V_0\left[\theta(-x)\frac{1}{\mathrm{e}^{-ax} - q} + \theta(x)\frac{1}{\mathrm{e}^{bx} - \widetilde{q}}\right], \tag{1}$$

where V_0 is the strength of the potential barrier, positive a, b, q and $\widetilde{q}$ parameters define the shape of the potential and $\theta(x)$ is the Heaviside step function. q and $\widetilde{q}$ are both less than unity. The shape of the asymmetric Hulthen potential is displayed in figure 1.

The structure of the paper will be as follows. In section 2, we exactly solve the DKP equation for spin-1 sector for an asymmetric form of the Hulthen potential. In section 3, we present the solutions of the scattering states in terms of the hypergeometric functions and derive the condition for the existence of the transmission resonances. Section 4 is devoted for the complete bound state solutions of the DKP equation for the asymmetric Hulthen potential well. We give our final remarks in section 5.

2. Asymmetric Hulthen potential in the DKP equation

The relativistic DKP equation [22] describes the nature of the massive vector (spin-1) and scalar (spin-0) bosons. Its covariant form in the presence of an external potential field is given by (in natural units $\hbar = c = 1$)

$$[\mathrm{i}\beta^{\mu}(\partial_{\mu} + \mathrm{i}eA_{\mu}) - m]\Psi_K(t, \vec{x}) = 0, \tag{2}$$

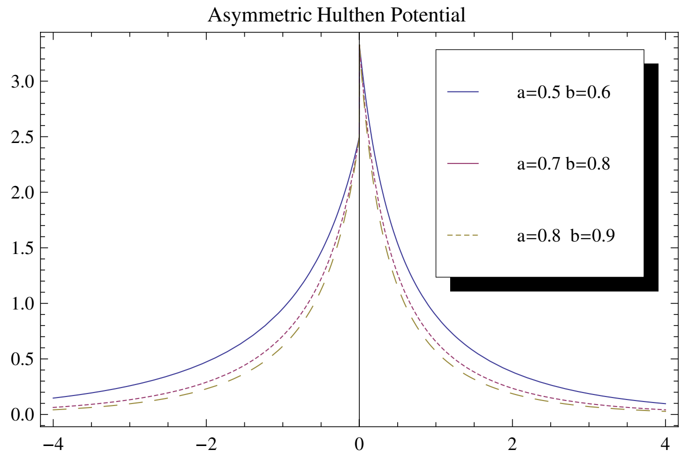

Figure 1. The shape of the asymmetric Hulthen potential for different values of the potential shape parameters a and b, with $V_0 = 1$, $q = 0.6$ and $\widetilde{q} = 0.7$.

where A_μ is the vector potential and m is the mass of the boson. The β^μ matrices satisfy the below relation:

$$\beta^\mu \beta^\nu \beta^\lambda + \beta^\lambda \beta^\nu \beta^\mu = g^{\mu\nu} \beta^\lambda + g^{\lambda\nu} \beta^\mu, \tag{3}$$

with $g^{\mu\nu} = \mathrm{diag}(+1, -1, -1, -1)$ metric tensor of the Minkowski spacetime. Beta matrices satisfying the DKP algebra have three irreducible representations of dimensions 1, 5 and 10. The wavefunction of the DKP equation, $\Psi_K(t, \overrightarrow{x})$, is 16-component and it remains invariant under the local Lorentz transformations. An appropriate linear transformation that takes the equation back to the its original form in the new co-ordinate system is acceptable for the requirement of the relativistic invariance of the theory. For a Lorentz transformation [23]: $x'^\mu = \Lambda^\mu_\nu x^\nu$

$$\Psi \rightarrow \Psi' = U\,(\Lambda)\,\Psi \tag{4}$$

$$U^{-1} \beta^\mu U = \Lambda^\mu_\nu \beta^\nu, \tag{5}$$

and for the case of general infinitesimal transformations $\Lambda^{\mu\nu} = \eta^{\mu\nu} + \omega^{\mu\nu}$ ($\omega^{\mu\nu} = -\omega^{\nu\mu}$) the infinitesimal Lorentz transformations are given as follows:

$$U = 1 + \tfrac{1}{2}\omega^{\mu\nu} S_{\mu\nu}, \qquad S_{\mu\nu} = [\beta_\mu, \beta_\nu]. \tag{6}$$

The particular spin-1 part of the DKP equation has been obtained [24, 25] from the quantization of a classical model in which the beta matrices are given by

$$\beta^\mu = \gamma^\mu \otimes I + I \otimes \gamma^\mu, \tag{7}$$

where γ^μ are the standard Dirac matrices. In this model spin-1 particle is considered as a particle system of two spin-1/2 particles with equal mass, instead of a single spin-1 particle. This is the second quantization approach of Schrödinger to the problem. In that case the wavefunction Ψ in equation (2) can be written in the form

$$\Psi = \Psi_K = \Psi_D \otimes \Psi_D, \tag{8}$$

where Ψ_K and Ψ_D are the wavefunctions of DKP and Dirac equations, respectively.

For the (1 + 1)-dimensional case, the Dirac gamma matrices in equation (7) are replaced [26] by the Pauli spin matrices σ^μ and the beta matrices are rewritten in the form

$$\beta^\mu = \sigma^\mu \otimes I + I \otimes \sigma^\mu, \tag{9}$$

and the DKP equation for the spin-1 sector takes the following form:

$$[\mathrm{i}(\sigma^\mu \otimes I + I \otimes \sigma^\mu)(\partial_\mu + \mathrm{i}eA_\mu) - m]\Psi_K = 0, \tag{10}$$

where Ψ_K^T is given by

$$\Psi_K^T = (\Psi_1 \quad \Psi_0 \quad \Psi_{\widetilde{0}} \quad \Psi_2).$$

We choose the representation of the Dirac gamma matrices as $\gamma^0 = \sigma^z$, $\gamma^1 = \mathrm{i}\sigma^x$. Then, the following four coupled differential equations are obtained from equation (10):

$$\begin{aligned}
&[2\,(\partial_0 + \mathrm{i}eA_0) + \mathrm{i}m]\,\Psi_1 + \mathrm{i}(\partial_1 + \mathrm{i}eA_1)(\Psi_0 + \Psi_{\widetilde{0}}) = 0\\
&(\partial_1 + \mathrm{i}eA_1)(\Psi_1 + \Psi_2) + m\Psi_0 = 0\\
&(\partial_1 + \mathrm{i}eA_1)(\Psi_1 + \Psi_2) + m\Psi_{\widetilde{0}} = 0\\
&[-2\,(\partial_0 + \mathrm{i}eA_0) + \mathrm{i}m]\,\Psi_2 + \mathrm{i}(\partial_1 + \mathrm{i}eA_1)(\Psi_0 + \Psi_{\widetilde{0}}) = 0.
\end{aligned} \tag{11}$$

After a simple algebra within these equations, a second-order differential equation is obtained:

$$[(\partial_1 + \mathrm{i}eA_1)^2 - (-\mathrm{i}E + \mathrm{i}eA_0)^2 - \widetilde{m}^2](\chi_1 + \chi_2) = 0, \tag{12}$$

where $\Psi(t,x) = \mathrm{e}^{-\mathrm{i}Et}\chi(x)$, $\chi^T = (\chi_1 \quad \chi_0 \quad \chi_{\widetilde{0}} \quad \chi_2)$ and $\widetilde{m} = \frac{m}{2}$. Exact solutions of equation (12) for the asymmetric Hulthen potential given by equation (1) can be obtained by defining two new variables: $y = q\mathrm{e}^{ax}$ for $x < 0$ and $z = \widetilde{q}\mathrm{e}^{-bx}$ for $x > 0$.

First, we consider the $x < 0$ case. Then, equation (12) takes the following form:

$$\left\{y^2\partial_y^2 + y\partial_y + \left(\frac{V_0}{aq}\right)^2\left(\frac{y}{1-y}\right)^2 + \frac{E^2 - \widetilde{m}^2}{a^2} - \frac{2V_0E}{a^2q}\left(\frac{y}{1-y}\right)\right\}(\chi_1 + \chi_2) = 0. \tag{13}$$

By setting $\chi_1(y) + \chi_2(y) = y^\mu(1-y)^\varepsilon f(y)$, the above equation reduces to the hypergeometric equation [27]:

$$y(1-y)\frac{\mathrm{d}^2 f}{\mathrm{d}y^2} + [1 + 2\mu - y\,(2\mu + 2\varepsilon + 1)]\,\frac{\mathrm{d}f}{\mathrm{d}y} + (\mu + \varepsilon + \beta)\,(\mu + \varepsilon - \beta)\,f(y) = 0, \tag{14}$$

where $\mu = \frac{\mathrm{i}\sqrt{E^2 - \widetilde{m}^2}}{a}$, $\varepsilon = \frac{1 + \sqrt{1 - 4(\frac{V_0}{aq})^2}}{2}$ and $\beta = \frac{\sqrt{\widetilde{m}^2 - (E + \frac{V_0}{q})^2}}{a}$. The solution of equation (14) is given in terms of hypergeometric functions [27]:

$$\begin{aligned}
f(y) = A\;{}_2F_1(\mu + \varepsilon - \beta, \mu + \varepsilon + \beta; 1 + 2\mu; y)\\
+ By^{-2\mu}\;{}_2F_1(-\mu + \varepsilon - \beta, -\mu + \varepsilon + \beta; 1 - 2\mu; y).
\end{aligned} \tag{15}$$

The left-hand side solutions are

$$\begin{aligned}
(\chi_1 + \chi_2)_L = Ay^\mu\,(1-y)^\varepsilon\;{}_2F_1(\mu + \varepsilon - \beta, \mu + \varepsilon + \beta; 1 + 2\mu; y)\\
+ By^{-\mu}\,(1-y)^\varepsilon\;{}_2F_1(-\mu + \varepsilon - \beta, -\mu + \varepsilon + \beta; 1 - 2\mu; y).
\end{aligned} \tag{16}$$

Now we consider the solutions for $x > 0$. In that case equation (12) becomes

$$\left\{z^2\partial_z^2 + z\partial_z + \left(\frac{V_0}{b\widetilde{q}}\right)^2\left(\frac{z}{1-z}\right)^2 + \frac{E^2 - \widetilde{m}^2}{b^2} - \frac{2V_0E}{b^2\widetilde{q}}\left(\frac{z}{1-z}\right)\right\}(\chi_1 + \chi_2) = 0. \tag{17}$$

By setting $\chi_1(z) + \chi_2(z) = z^{\widetilde{\mu}}(1-z)^{-\widetilde{\varepsilon}} g(z)$, equation (17) gives the hypergeometric equation

$$z(1-z)\frac{\mathrm{d}^2 g}{\mathrm{d}z^2} + [1 + 2\widetilde{\mu} - z(2\widetilde{\mu} - 2\widetilde{\varepsilon} + 1)]\frac{\mathrm{d}g}{\mathrm{d}z} + (\widetilde{\mu} - \widetilde{\varepsilon} + \widetilde{\beta})(\widetilde{\mu} - \widetilde{\varepsilon} - \widetilde{\beta})g(z) = 0, \tag{18}$$

where $\widetilde{\mu} = \frac{\mathrm{i}\sqrt{E^2 - \widetilde{m}^2}}{b}$, $\widetilde{\varepsilon} = \frac{-1+\sqrt{1-4(\frac{V_0}{bq})^2}}{2}$ and $\widetilde{\beta} = \frac{\sqrt{\widetilde{m}^2 - (E + \frac{V_0}{q})^2}}{b}$.

Solution of this equation will be

$$g(z) = C\,{}_2F_1(\widetilde{\mu} - \widetilde{\varepsilon} - \widetilde{\beta}, \widetilde{\mu} - \widetilde{\varepsilon} + \widetilde{\beta}; 1 + 2\widetilde{\mu}; z) + Dz^{-2\widetilde{\mu}}\,{}_2F_1(-\widetilde{\mu} - \widetilde{\varepsilon} - \widetilde{\beta}, -\widetilde{\mu} - \widetilde{\varepsilon} + \widetilde{\beta}; 1 - 2\widetilde{\mu}; z), \tag{19}$$

and the right-hand side solutions are

$$(\chi_1 + \chi_2)_R = Cz^{\widetilde{\mu}}(1-z)^{-\widetilde{\varepsilon}}\,{}_2F_1(\widetilde{\mu} - \widetilde{\varepsilon} - \widetilde{\beta}, \widetilde{\mu} - \widetilde{\varepsilon} + \widetilde{\beta}; 1 + 2\widetilde{\mu}; z) + Dz^{-\widetilde{\mu}}(1-z)^{-\widetilde{\varepsilon}}\,{}_2F_1(-\widetilde{\mu} - \widetilde{\varepsilon} - \widetilde{\beta}, -\widetilde{\mu} - \widetilde{\varepsilon} + \widetilde{\beta}; 1 - 2\widetilde{\mu}; z). \tag{20}$$

The solutions obtained so far are for the linear combination of the first and last components of the wavefunction. The other components can be obtained with the help of the following equations:

$$\Psi_0 = -\frac{1}{m}(\partial_1 + \mathrm{i}eA_1)(\Psi_1 + \Psi_2), \tag{21}$$

$$(\Psi_1 - \Psi_2) = \frac{2\mathrm{i}}{m}(\partial_0 + \mathrm{i}eA_0)(\Psi_1 + \Psi_2), \tag{22}$$

$$\frac{\mathrm{d}\,{}_2F_1(a, b; c; x)}{\mathrm{d}x} = \frac{ab}{c}\,{}_2F_1(a+1, b+1; c+1; x). \tag{23}$$

3. Scattering states

From now on, we can discuss the reflection and transmission through the potential barrier given by equation (1) where $V_0 > 0$. For the physical solutions of the system, we should ensure that our solutions satisfy the appropriate boundary conditions. The asymmetric Hulthen potential vanishes as $x \to -\infty$, thus the incoming wave should behave as a plane wave for $x \to -\infty$. Taking into account that as $x \to -\infty$ then $y \to 0$ and by considering the asymptotic behavior of the hypergeometric functions [27] as $t \to 0$:

$${}_2F_1(a, b; c; t) \longrightarrow 1, \tag{24}$$

we obtain the complete solutions of the incoming wave and its asymptotic form for very large values of x as follows:

$$\Psi_{\mathrm{inc.}} = y^{\mu}(1-y)^{\varepsilon} \times \begin{bmatrix} \left\{\frac{1}{m}[E - \frac{V_0}{q}\left(\frac{y}{1-y}\right)] + \frac{1}{2}\right\}F_{\mathrm{inc.}} \\ \frac{a}{m}\left\{\left[\frac{-\mu + y(\mu+\varepsilon)}{1-y}\right]F_{\mathrm{inc.}} + \left[\frac{\beta^2 - (\mu+\varepsilon)^2}{1+2\mu}\right]y\widetilde{F}_{\mathrm{inc.}}\right\} \\ \frac{a}{m}\left\{\left[\frac{-\mu + y(\mu+\varepsilon)}{1-y}\right]F_{\mathrm{inc.}} + \left[\frac{\beta^2 - (\mu+\varepsilon)^2}{1+2\mu}\right]y\widetilde{F}_{\mathrm{inc.}}\right\} \\ \left\{\frac{-1}{m}[E - \frac{V_0}{q}\left(\frac{y}{1-y}\right)] + \frac{1}{2}\right\}F_{\mathrm{inc.}} \end{bmatrix} \tag{25}$$

$$\Psi_{\mathrm{inc.}} \longrightarrow q^{\mathrm{i}k/a}\,\mathrm{e}^{\mathrm{i}kx} \begin{bmatrix} \left(\frac{E}{m} + \frac{1}{2}\right) \\ \frac{-\mathrm{i}k}{m} \\ \frac{-\mathrm{i}k}{m} \\ \left(\frac{-E}{m} + \frac{1}{2}\right) \end{bmatrix}, \tag{26}$$

where we used the below definitions

$$F_{\text{inc.}} = {}_2F_1(\mu + \varepsilon - \beta, \mu + \varepsilon + \beta; 1 + 2\mu; y)$$
$$\widetilde{F}_{\text{inc.}} = {}_2F_1(\mu + \varepsilon - \beta + 1, \mu + \varepsilon + \beta + 1; 2 + 2\mu; y)$$
$$k = \sqrt{E^2 - \widetilde{m}^2}.$$

As it is seen from the asymptotic behavior, the incoming wave has the required momentum direction.

The reflected wave can be written by replacing μ with $-\mu$ in the incoming wave. Then we have the reflected wave and its asymptotic form for very large values of x as follows:

$$\Psi_{\text{ref.}} = y^{-\mu}(1-y)^{\varepsilon} \times \begin{bmatrix} \left\{\frac{1}{m}\left[E - \frac{V_0}{q}\left(\frac{y}{1-y}\right)\right] + \frac{1}{2}\right\} F_{\text{ref.}} \\ \frac{a}{m}\left\{\left[\frac{\mu+y(-\mu+\varepsilon)}{1-y}\right]F_{\text{ref.}} + \left[\frac{\beta^2-(-\mu+\varepsilon)^2}{1-2\mu}\right] y\widetilde{F}_{\text{ref.}}\right\} \\ \frac{a}{m}\left\{\left[\frac{\mu+y(-\mu+\varepsilon)}{1-y}\right]F_{\text{ref.}} + \left[\frac{\beta^2-(-\mu+\varepsilon)^2}{1-2\mu}\right] y\widetilde{F}_{\text{ref.}}\right\} \\ \left\{\frac{-1}{m}\left[E - \frac{V_0}{q}\left(\frac{y}{1-y}\right)\right] + \frac{1}{2}\right\} F_{\text{ref.}} \end{bmatrix}, \tag{27}$$

$$\Psi_{\text{ref.}} \longrightarrow q^{-\mathrm{i}k/a}\,\mathrm{e}^{-\mathrm{i}kx} \begin{bmatrix} \left(\frac{E}{m} + \frac{1}{2}\right) \\ \frac{\mathrm{i}k}{m} \\ \frac{\mathrm{i}k}{m} \\ \left(\frac{-E}{m} + \frac{1}{2}\right) \end{bmatrix}, \tag{28}$$

where we used the definitions

$$F_{\text{ref.}} = {}_2F_1(-\mu + \varepsilon - \beta, -\mu + \varepsilon + \beta; 1 - 2\mu; y)$$
$$\widetilde{F}_{\text{ref.}} = {}_2F_1(-\mu + \varepsilon - \beta + 1, -\mu + \varepsilon + \beta + 1; 2 - 2\mu; y).$$

Finally, the transmitted wave and its asymptotic form for very large values of x are obtained as

$$\Psi_{\text{trans.}} = z^{\widetilde{\mu}}(1-z)^{-\widetilde{\varepsilon}} \times \begin{bmatrix} \left\{\frac{1}{m}\left[E - \frac{V_0}{\widetilde{q}}\left(\frac{z}{1-z}\right)\right] + \frac{1}{2}\right\} F_{\text{trans.}} \\ \frac{b}{m}\left\{\left[\frac{-\widetilde{\mu}+z(\widetilde{\mu}+\widetilde{\varepsilon})}{1-z}\right]F_{\text{trans.}} + \left[\frac{-\widetilde{\beta}^2+(\widetilde{\mu}+\widetilde{\varepsilon})^2}{1-2\widetilde{\mu}}\right] z\widetilde{F}_{\text{trans.}}\right\} \\ \frac{b}{m}\left\{\left[\frac{-\widetilde{\mu}+z(\widetilde{\mu}+\widetilde{\varepsilon})}{1-z}\right]F_{\text{trans.}} + \left[\frac{-\widetilde{\beta}^2+(\widetilde{\mu}+\widetilde{\varepsilon})^2}{1-2\widetilde{\mu}}\right] z\widetilde{F}_{\text{trans.}}\right\} \\ \left\{\frac{-1}{m}\left[E - \frac{V_0}{\widetilde{q}}\left(\frac{z}{1-z}\right)\right] + \frac{1}{2}\right\} F_{\text{trans.}} \end{bmatrix}, \tag{29}$$

$$\Psi_{\text{trans.}} \longrightarrow \widetilde{q}^{-\mathrm{i}k/b}\,\mathrm{e}^{\mathrm{i}kx} \begin{bmatrix} \left(\frac{E}{m} + \frac{1}{2}\right) \\ \frac{-\mathrm{i}k}{m} \\ \frac{-\mathrm{i}k}{m} \\ \left(\frac{-E}{m} + \frac{1}{2}\right) \end{bmatrix}, \tag{30}$$

where

$$F_{\text{trans.}} = {}_2F_1(-\widetilde{\mu} - \widetilde{\varepsilon} - \widetilde{\beta}, -\widetilde{\mu} - \widetilde{\varepsilon} + \widetilde{\beta}; 1 - 2\widetilde{\mu}; z)$$
$$\widetilde{F}_{\text{trans.}} = {}_2F_1(-\widetilde{\mu} - \widetilde{\varepsilon} - \widetilde{\beta} + 1, -\widetilde{\mu} - \widetilde{\varepsilon} + \widetilde{\beta} + 1; 2 - 2\widetilde{\mu}; z).$$

We can discuss the transmission resonances in consideration of determining the reflection and transmission coefficients. For this purpose we use the continuity of the DKP wavefunction at $x = 0$:

$$N_1\Psi_{\text{inc.}}(x=0) + N_2\Psi_{\text{ref.}}(x=0) = N_3\Psi_{\text{trans.}}(x=0). \tag{31}$$

Table 1. Table for the constants defined in the calculations of T and R.

$\alpha_1 = \frac{1}{m}\left[E - \frac{V_0}{1-q}\right] + \frac{1}{2}$	$\beta_1 = \frac{1}{m}\left[E - \frac{V_0}{1-\widetilde{q}}\right] + \frac{1}{2}$
$\alpha_0 = \frac{a}{m}\left[\frac{-\mu+q(\mu+\varepsilon)}{1-q}\right]$	$\beta_0 = \frac{b}{m}\left[\frac{-\widetilde{\mu}+\widetilde{q}(\widetilde{\mu}+\widetilde{\varepsilon})}{1-\widetilde{q}}\right]$
$\widetilde{\alpha}_0 = \frac{aq}{m}\left[\frac{\beta^2-(\mu+\varepsilon)^2}{1+2\mu}\right]$	$\widetilde{\beta}_0 = \frac{b\widetilde{q}}{m}\left[\frac{-\widetilde{\beta}^2+(\widetilde{\mu}+\widetilde{\varepsilon})^2}{1-2\widetilde{\mu}}\right]$
$\alpha_2 = \frac{-1}{m}\left[E - \frac{V_0}{1-q}\right] + \frac{1}{2}$	$\beta_2 = \frac{-1}{m}\left[E - \frac{V_0}{1-\widetilde{q}}\right] + \frac{1}{2}$
$\gamma_0 = \frac{a}{m}\left[\frac{\mu+q(-\mu+\varepsilon)}{1-q}\right]$	$\widetilde{\gamma}_0 = \frac{aq}{m}\left[\frac{\beta^2-(-\mu+\varepsilon)^2}{1-2\mu}\right]$

The relationship between N_i coefficients can be computed with the help of the above condition. Since the four-vector current density must be conserved between the negative infinity and the positive infinity, the reflection and transmission coefficients can be determined by using the conservation of the current density.

The four-vector current density for the DKP equation is given by $j^\mu = \overline{\Psi}\beta^\mu\Psi$ where $\overline{\Psi} = \Psi^\dagger\left(\gamma^0 \otimes \gamma^0\right)$. Thus, the reflection and transmission coefficients are defined in terms of the incident, reflected and transmitted currents and can be computed by

$$T = \left|\frac{j_{\text{trans.}}}{j_{\text{inc.}}}\right| = \left|\frac{N_3}{N_1}\right|^2, \tag{32}$$

$$R = \left|\frac{j_{\text{ref.}}}{j_{\text{inc.}}}\right| = \left|\frac{N_2}{N_1}\right|^2. \tag{33}$$

The continuity condition of the DKP wave at the boundary, at $x = 0$, gives

$$\begin{aligned}\frac{N_3}{N_1} &= \frac{q^\mu(1-q)^\varepsilon}{\widetilde{q}^{-\widetilde{\mu}}(1-\widetilde{q})^{-\widetilde{\varepsilon}}}\\ &\quad\times\left\{\frac{F_{\text{inc.}}(q)[\gamma_0 F_{\text{ref.}}(q) + \widetilde{\gamma}_0\widetilde{F}_{\text{ref.}}(q)] - F_{\text{ref.}}(q)[\alpha_0 F_{\text{inc.}}(q) + \widetilde{\alpha}_0\widetilde{F}_{\text{inc.}}(q)]}{\frac{\beta_1}{\alpha_1}F_{\text{trans.}}(\widetilde{q})[\gamma_0 F_{\text{ref.}}(q) + \widetilde{\gamma}_0\widetilde{F}_{\text{ref.}}(q)] - F_{\text{ref.}}(q)[\beta_0 F_{\text{trans.}}(\widetilde{q}) + \widetilde{\beta}_0\widetilde{F}_{\text{trans.}}(\widetilde{q})]}\right\},\\ \frac{N_2}{N_1} &= \frac{q^{2\mu}}{F_{\text{ref.}}(q)}\left[\frac{N_3}{N_1}\frac{\widetilde{q}^{-\widetilde{\mu}}(1-\widetilde{q})^{-\widetilde{\varepsilon}}}{q^\mu(1-q)^\varepsilon}\frac{\beta_2}{\alpha_2}F_{\text{trans.}}(\widetilde{q}) - F_{\text{inc.}}(q)\right],\end{aligned} \tag{34}$$

where the use of below definitions are made

$$\begin{aligned}F_{\text{inc.}}(q) &= {}_2F_1(\mu+\varepsilon-\beta, \mu+\varepsilon+\beta; 1+2\mu; q),\\ \widetilde{F}_{\text{inc.}}(q) &= {}_2F_1(\mu+\varepsilon-\beta+1, \mu+\varepsilon+\beta+1; 2+2\mu; q),\\ F_{\text{ref.}}(q) &= {}_2F_1(-\mu+\varepsilon-\beta, -\mu+\varepsilon+\beta; 1-2\mu; q),\\ \widetilde{F}_{\text{ref.}}(q) &= {}_2F_1(-\mu+\varepsilon-\beta+1, -\mu+\varepsilon+\beta+1; 2-2\mu; q),\\ F_{\text{trans.}}(\widetilde{q}) &= {}_2F_1(-\widetilde{\mu}-\widetilde{\varepsilon}-\widetilde{\beta}, -\widetilde{\mu}-\widetilde{\varepsilon}+\widetilde{\beta}; 1-2\widetilde{\mu}; \widetilde{q}),\\ \widetilde{F}_{\text{trans.}}(\widetilde{q}) &= {}_2F_1(-\widetilde{\mu}-\widetilde{\varepsilon}-\widetilde{\beta}+1, -\widetilde{\mu}-\widetilde{\varepsilon}+\widetilde{\beta}+1; 2-2\widetilde{\mu}; \widetilde{q}).\end{aligned}$$

The constants α, β and γ defined in the calculations of the R and T are listed in table 1. By using equations (34) in equations (32) and (33) we can check whether the unitarity condition, $R + T = 1$, is satisfied or not. We have done this by using a mathematical software package. For example, if we take $a = b = 0.7, q = \widetilde{q} = 0.8, m = 1$ and $V_0 = 4$, the $R + T = 1$ unitarity condition is certainly satisfied.

Now we can discuss the condition for the existence of the transmission resonances. The condition for the transmission resonances to appear is equal to the set $R = 0$. Then, equaling $\frac{N_2}{N_1}$ to zero we obtain the condition for the transmission resonances:

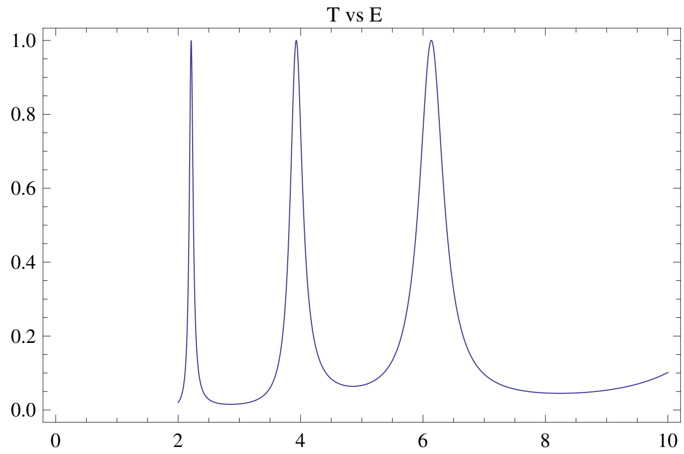

Figure 2. Transmission coefficient versus energy for $a = b = 0.7, q = \widetilde{q} = 0.8, m = 1$ and $V_0 = 4$.

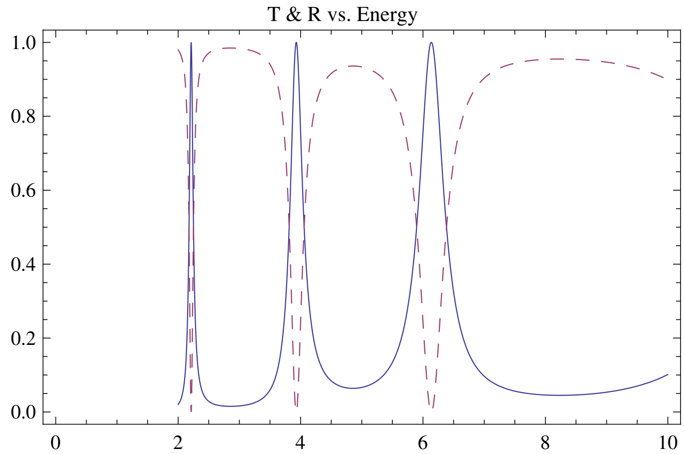

Figure 3. Unitarity condition, $R + T = 1$, is displayed in the plot for parameters $a = b = 0.7, q = \widetilde{q} = 0.8, m = 1$ and $V_0 = 4$. Solid and dashed lines correspond to the T and R, respectively.

$$\{F_{\text{inc.}}(q)F_{\text{trans.}}(\widetilde{q})F_{\text{ref.}}(q)\left[\left(\frac{\beta_2}{\alpha_2} - \frac{\beta_1}{\alpha_1}\right)\gamma_0 - \frac{\beta_2}{\alpha_2}\alpha_0\right] + F_{\text{inc.}}(q)F_{\text{trans.}}(\widetilde{q})\widetilde{F}_{\text{ref.}}(q)\left(\frac{\beta_2}{\alpha_2} - \frac{\beta_1}{\alpha_1}\right)\widetilde{\gamma}_0 - F_{\text{ref.}}(q)F_{\text{trans.}}(\widetilde{q})\widetilde{F}_{\text{inc.}}(q)\frac{\beta_2}{\alpha_2}\widetilde{\alpha}_0 + F_{\text{inc.}}(q)F_{\text{ref.}}(q)\widetilde{F}_{\text{trans.}}(\widetilde{q})\widetilde{\beta}_0\} = 0. \quad (35)$$

Figures 1 and 3 explicitly show the existence of the transmission resonances and the unitarity condition, respectively. It is interesting to discuss the transmission resonance phenomena for the solutions of the DKP equation for the Hulthen potential. The dependence of the transmission coefficient on the energy of the vector particle varies by different values of the q and $\widetilde{q}$ parameters. This can be determined numerically. Figure 4 shows how the transmission coefficient depends on the energy of the particle. It is seen that the intensity and the widths of the resonance peaks are sensitive to the value of $\widetilde{q}$. As $\widetilde{q}$ increases, resonance peaks become

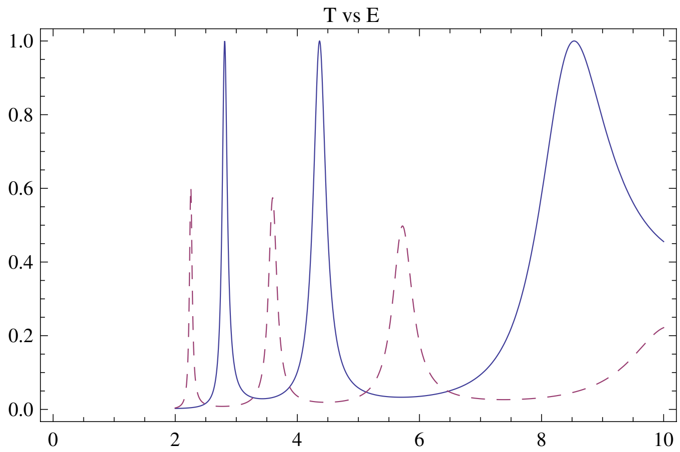

Figure 4. Transmission coefficient versus energy for $a = b = 0.6, m = 1$ and $V_0 = 4$. The solid line represents the case for $q = \widetilde{q} = 0.7$ and the dashed line corresponds to the case for $q = 0.7, \widetilde{q} = 0.8$.

shorter and narrower. Figure 5 indicates that, although the shape parameters a and b take different values, we still have transmission resonances as $q = \widetilde{q}$. Resonance peaks appear at smaller values of particle energy in cases $a < b$. Finally, figures 6 and 7 show the dependence of the transmission resonances on the potential strength parameter V_0. Increasing the $\widetilde{q}$ value leads to more transmission resonance peaks, in cases where the shape parameters have the same values. In figure 7, it is seen that the same number of transmission resonance peaks occur for varying a and b, whereas q and $\widetilde{q}$ have equal values. The transmission resonance peaks show a periodical behavior as the energy increases. Based on the figures, we can also conclude that both symmetric and asymmetric Hulthen potentials are completely penetrable. The outgoing wave is shifted by δ in its phase relative to the incoming wave which can be defined in terms of the transmission coefficient by

$$\frac{N_3}{N_1} = \sqrt{T}\,\mathrm{e}^{-\mathrm{i}\delta}. \tag{36}$$

An analytical expression for the phase shift can be found as follows by using $\frac{N_3}{N_1}$ from equation (34):

$$\delta = -\mathrm{i}\ln\left[\sqrt{T}\frac{N_1}{N_3}\right]. \tag{37}$$

4. Bound states

In this section, we consider the solutions for bound states, $|E| < m$. In that case, the Hulthen potential barrier becomes an attractive potential well for $V_0 < 0$:

$$eA_0 = V(x) = -V_0\left[\theta(-x)\frac{1}{\mathrm{e}^{-ax} - q} + \theta(x)\frac{1}{\mathrm{e}^{bx} - \widetilde{q}}\right]. \tag{38}$$

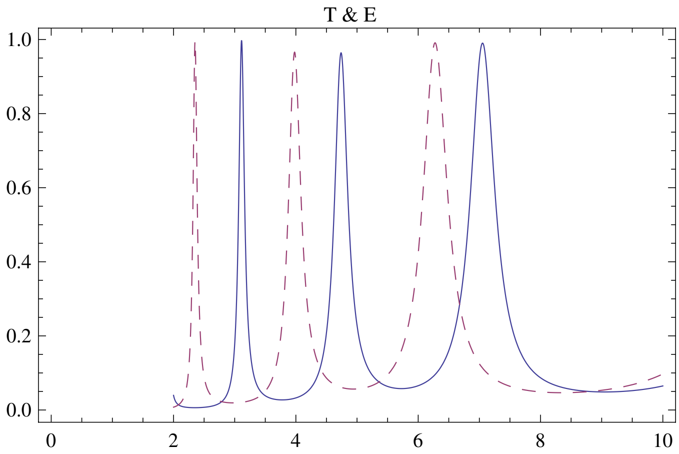

Figure 5. Transmission coefficient versus energy for $q = \widetilde{q} = 0.8$, $m = 1$ and $V_0 = 4$. The solid line is for $a = 0.7$, $b = 0.5$ and the dashed line is for $a = 0.6$, $b = 0.8$.

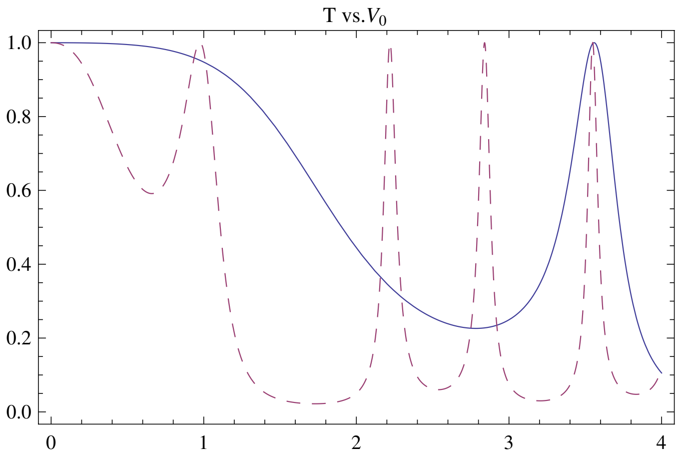

Figure 6. Transmission coefficient versus potential strength for $a = b = 0.7$, $m = 1$ and $E = 4$. The solid line is for $q = \widetilde{q} = 0.5$ and the dashed line is for $q = 0.5$ and $\widetilde{q} = 0.9$.

Considering equation (13) we see that the equation to be solved for the bound states for $x < 0$ is

$$\left\{ y^2\partial_y^2 + y\partial_y + \left(\frac{-V_0}{aq}\right)^2\left(\frac{y}{1-y}\right)^2 + \frac{E^2-\widetilde{m}^2}{a^2} + \frac{2V_0E}{a^2q}\left(\frac{y}{1-y}\right)\right\}(\chi_1+\chi_2) = 0. \tag{39}$$

By setting $\chi_1(y) + \chi_2(y) = y^{\xi}(1-y)^{\varepsilon}\,w(y)$, the above equation reduces to the hypergeometric equation

$$y(1-y)\frac{\mathrm{d}^2w}{\mathrm{d}y^2} + [1+2\xi - y\,(2\xi+2\varepsilon+1)]\frac{\mathrm{d}w}{\mathrm{d}y} + (\xi+\varepsilon+\nu)\,(\xi+\varepsilon-\nu)\,w(y) = 0, \tag{40}$$

where $\xi = \frac{\sqrt{\widetilde{m}^2-E^2}}{a}$, $\varepsilon = \frac{1+\sqrt{1-4(\frac{V_0}{aq})^2}}{2}$ and $\nu = \frac{\sqrt{\widetilde{m}^2-(E-\frac{V_0}{q})^2}}{a}$.

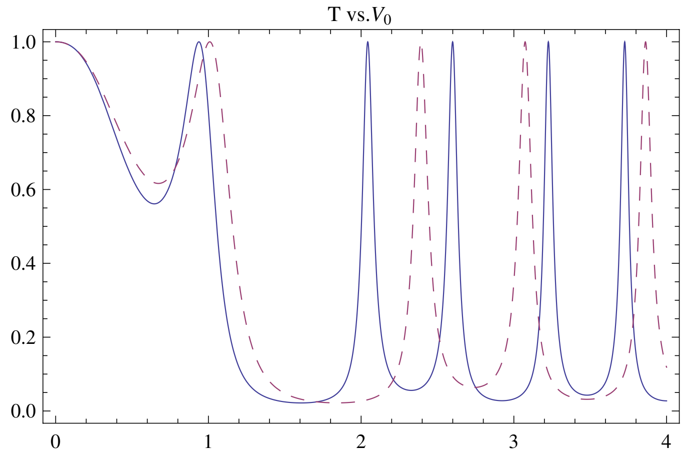

Figure 7. Transmission coefficient versus potential strength for $q = \widetilde{q} = 0.9$, $m = 1$ and $E = 4$. The solid line is for $a = b = 0.6$ and the dashed line is for $a = 0.6$, $b = 0.8$.

The solution of equation (38) is given by the hypergeometric function as

$$w(y) = C_1\, {}_2F_1(\xi + \varepsilon - \nu, \xi + \varepsilon + \nu; 1 + 2\xi; y) + C_2 y^{-2\xi}\, {}_2F_1(-\xi + \varepsilon - \nu, -\xi + \varepsilon + \nu; 1 - 2\xi; y). \tag{41}$$

Then complete bound state solutions of the DKP equation for $x < 0$ are obtained as follows:

$$\Psi_L = y^{\xi}(1-y)^{\varepsilon} \begin{bmatrix} \left\{\frac{1}{m}\left[E + \frac{V_0}{q}\left(\frac{y}{1-y}\right)\right] + \frac{1}{2}\right\} F_L \\ \frac{a}{m}\left\{\left[\frac{-\xi + y(\xi+\varepsilon)}{1-y}\right] F_L + \left[\frac{\nu^2 - (\xi+\varepsilon)^2}{1+2\xi}\right] y \widetilde{F}_L\right\} \\ \frac{a}{m}\left\{\left[\frac{-\xi + y(\xi+\varepsilon)}{1-y}\right] F_L + \left[\frac{\nu^2 - (\xi+\varepsilon)^2}{1+2\xi}\right] y \widetilde{F}_L\right\} \\ \left\{\frac{-1}{m}\left[E + \frac{V_0}{q}\left(\frac{y}{1-y}\right)\right] + \frac{1}{2}\right\} F_L \end{bmatrix}, \tag{42}$$

where

$$F_L = {}_2F_1(\xi + \varepsilon - \nu, \xi + \varepsilon + \nu; 1 + 2\xi; y)$$
$$\widetilde{F}_L = {}_2F_1(\xi + \varepsilon - \nu + 1, \xi + \varepsilon + \nu + 1; 2 + 2\xi; y).$$

Next, we study the bound state solutions of DKP equation for $x > 0$. Equation (17) takes the following form:

$$\left\{z^2\partial_z^2 + z\partial_z + \left(\frac{-V_0}{b\widetilde{q}}\right)^2\left(\frac{z}{1-z}\right)^2 + \frac{E^2 - \widetilde{m}^2}{b^2} + \frac{2V_0E}{b^2\widetilde{q}}\left(\frac{z}{1-z}\right)\right\}(\chi_1 + \chi_2) = 0 \tag{43}$$

and setting $\chi_1(z) + \chi_2(z) = z^{\widetilde{\xi}}(1-z)^{-\widetilde{\varepsilon}} s(z)$, equation (41) gives the hypergeometric equation

$$z(1-z)\frac{\mathrm{d}^2 s}{\mathrm{d}z^2} + [1 + 2\widetilde{\xi} - z(2\widetilde{\xi} - 2\widetilde{\varepsilon} + 1)]\frac{\mathrm{d}s}{\mathrm{d}z} + (\widetilde{\xi} - \widetilde{\varepsilon} + \widetilde{\nu})(\widetilde{\xi} - \widetilde{\varepsilon} - \widetilde{\nu})s(z) = 0, \tag{44}$$

where $\widetilde{\xi} = \frac{\sqrt{\widetilde{m}^2 - E^2}}{b}$, $\widetilde{\varepsilon} = \frac{1 + \sqrt{1 - 4\left(\frac{V_0}{b\widetilde{q}}\right)^2}}{2}$ and $\widetilde{\nu} = \frac{\sqrt{\widetilde{m}^2 - \left(E - \frac{V_0}{\widetilde{q}}\right)^2}}{b}$.

The solution of this equation is

$$s(z) = C_3\, {}_2F_1(\widetilde{\mu} - \widetilde{\varepsilon} - \widetilde{\beta}, \widetilde{\mu} - \widetilde{\varepsilon} + \widetilde{\beta}; 1 + 2\widetilde{\mu}; z) + C_4 z^{-2\widetilde{\mu}}\, {}_2F_1(-\widetilde{\mu} - \widetilde{\varepsilon} - \widetilde{\beta}, -\widetilde{\mu} - \widetilde{\varepsilon} + \widetilde{\beta}; 1 - 2\widetilde{\mu}; z). \tag{45}$$

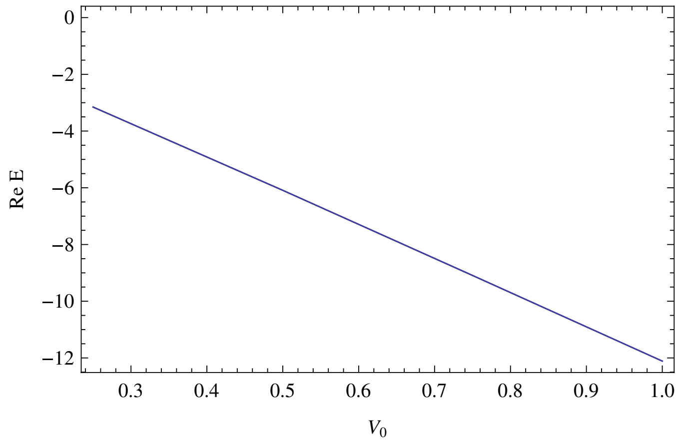

Figure 8. The dependence of the real part of the first bound state energy on the potential strength for $a = 0.4$, $b = 0.5$, $m = 1$, $q = 0.1$ and $\widetilde{q} = 0.15$.

Then, the bound state solutions of the DKP equation for $x > 0$ are

$$\Psi_R = z^{\widetilde{\xi}}\,(1-z)^{-\widetilde{\varepsilon}}\begin{bmatrix} \left\{\frac{1}{m}\left[E+\frac{V_0}{\widetilde{q}}\left(\frac{z}{1-z}\right)\right]+\frac{1}{2}\right\}F_R \\ \frac{b}{m}\left\{\left[\frac{-\widetilde{\xi}+z(\widetilde{\xi}+\widetilde{\varepsilon})}{1-z}\right]F_R+\left[\frac{-\widetilde{\nu}^2+(\widetilde{\xi}+\widetilde{\varepsilon})^2}{1-2\widetilde{\xi}}\right]z\widetilde{F}_R\right\} \\ \frac{b}{m}\left\{\left[\frac{-\widetilde{\xi}+z(\widetilde{\xi}+\widetilde{\varepsilon})}{1-z}\right]F_R+\left[\frac{-\widetilde{\nu}^2+(\widetilde{\xi}+\widetilde{\varepsilon})^2}{1-2\widetilde{\xi}}\right]z\widetilde{F}_R\right\} \\ \left\{\frac{-1}{m}\left[E+\frac{V_0}{\widetilde{q}}\left(\frac{z}{1-z}\right)\right]+\frac{1}{2}\right\}F_R \end{bmatrix}, \tag{46}$$

where

$$\begin{aligned} F_R &= {}_2F_1(-\widetilde{\xi}-\widetilde{\varepsilon}-\widetilde{\nu}, -\widetilde{\xi}-\widetilde{\varepsilon}+\widetilde{\nu}; 1-2\widetilde{\xi}; z) \\ \widetilde{F}_R &= {}_2F_1(-\widetilde{\xi}-\widetilde{\varepsilon}-\widetilde{\nu}+1, -\widetilde{\xi}-\widetilde{\varepsilon}+\widetilde{\nu}+1; 2-2\widetilde{\xi}; z). \end{aligned}$$

The bound state energy eigenvalues are found by requiring that the right- and left-hand side wavefunctions must be matched at $x = 0$:

$$\begin{aligned} &\left[\frac{1}{m}\left(E+\frac{V_0}{1-\widetilde{q}}\right)+\frac{1}{2}\right]F_R(\widetilde{q})\left[\frac{a}{m}\left(\frac{-\xi+q(\xi+\varepsilon)}{1-q}\right)F_L(q)+\frac{aq}{m}\left(\frac{\nu^2-(\xi+\varepsilon)^2}{1+2\xi}\right)\widetilde{F_L}(q)\right] \\ &\quad -\left[\frac{1}{m}\left(E+\frac{V_0}{1-q}\right)+\frac{1}{2}\right]F_L(q)\left[\frac{b}{m}\left(\frac{-\widetilde{\xi}+\widetilde{q}(\widetilde{\xi}+\widetilde{\varepsilon})}{1-\widetilde{q}}\right)F_R(\widetilde{q})\right. \\ &\quad \left. +\frac{b\widetilde{q}}{m}\left(\frac{-\widetilde{\nu}^2+(\widetilde{\xi}+\widetilde{\varepsilon})^2}{1-2\widetilde{\xi}}\right)\widetilde{F_R}(\widetilde{q})\right]=0, \end{aligned} \tag{47}$$

where below definitions are made:

$$\begin{aligned} F_L(q) &= {}_2F_1(\xi+\varepsilon-\nu, \xi+\varepsilon+\nu; 1+2\xi; q) \\ \widetilde{F_L}(q) &= {}_2F_1(\xi+\varepsilon-\nu+1, \xi+\varepsilon+\nu+1; 2+2\xi; q) \\ F_R(\widetilde{q}) &= {}_2F_1(-\widetilde{\xi}-\widetilde{\varepsilon}-\widetilde{\nu}, -\widetilde{\xi}-\widetilde{\varepsilon}+\widetilde{\nu}; 1-2\widetilde{\xi}; \widetilde{q}) \\ \widetilde{F_R}(\widetilde{q}) &= {}_2F_1(-\widetilde{\xi}-\widetilde{\varepsilon}-\widetilde{\nu}+1, -\widetilde{\xi}-\widetilde{\varepsilon}+\widetilde{\nu}+1; 2-2\widetilde{\xi}; \widetilde{q}). \end{aligned}$$

Using this equation, one can determine the energy eigenvalues of the bound states numerically. Following this approach, we illustrate the relation between energy and the potential strength in figure 8. If the potential well strengthens, then the bound state energy of the system decreases and as this energy becomes equal to the negative mass of the particle the bound states merge with the negative energy continuum. In that case, the potential takes its critical value and is called the supercritical potential [6]. The value of the supercritical potential can be obtained by taking $E \rightarrow -m$ in equation (45), which leads $\xi \rightarrow 0$ and $\widetilde{\xi} \rightarrow 0$. The critical potential is found to be $V = V_0 = 0.268\,019$.

5. Conclusion

We obtained the solutions of the DKP equation for the general form of the asymmetric Hulthen potential which includes various types of the physical potentials. Quantum effects related to these potentials are widely discussed in the literature for Klein–Gordon and Dirac equations. The DKP equation can be said to be less studied compared to them. Especially, transmission resonance phenomena in the DKP equation have not been examined yet.

In this study, we solved exactly the DKP equation for asymmetric Hulthen potential. The main concern of this study is to verify that whether the transmission resonance phenomena in the DKP equation occur only for the symmetric potentials, as suggested for the Dirac equation in the [14], or not. For this purpose we obtained scattering and bound state solutions and the condition for the existence of the transmission resonances in the DKP equation. We showed that these phenomena occur not only for symmetric potentials but also for the asymmetric potentials as displayed in the figure 7. The asymmetry of the Hulthen potential results from different values of a, b, q and $\widetilde{q}$ shape parameters. If we set $a = b$ and $q = \widetilde{q}$, then the transmission resonances occur periodically. For unequal values of a and b, the periodical occurrence of the transmission resonances, as displayed in figure 2, yields a sequential occurrence behavior, as in the figure 5. In this figure, the first peak corresponds to a transmission resonance while the second peak has an intensity less than unity. These results show that the transmission resonances appearing for different external potentials depend on their forms. Therefore, there can be no generalization for the DKP equation that requires the transmission coefficient has to be always less than unity for any asymmetric potential. We analyzed the dependence of the transmission coefficient on the energy of the particle and the strength of the potential barrier.

References

[1] Newton R G 1982 *Scattering Theory of Waves and Particles* (Berlin: Springer)
[2] Senn P 1988 *Am. J. Phys.* **56** 916–21
[3] Bianchi M S 1994 *J. Math. Phys.* **35** 2719–33
[4] Ballentine L E 1998 *Quantum Mechanics A Modern Development* (Singapore: World Scientific)
Morse P M and Feshbach H 1953 *Methods of Theoretical Physics, Part II* (New York: McGraw-Hill)
[5] Bohm D 1951 *Quantum Mechanics* (New York: Prentice-Hall)
[6] Dombey N, Kennedy P and Calogeracos A 2003 *Phys. Rev. Lett.* **85** 1787–90
[7] Kennedy P and Dombey N 2002 *J. Phys. A: Math. Gen.* **35** 6645–57
[8] Kennedy P 2002 *J. Phys. A: Math. Gen.* **35** 689–98
[9] Villalba V M and Greiner W 2003 *Phys. Rev.* A **67** 052707-1
[10] Rojas C and Villalba V M 2005 *Phys. Rev.* A **71** 052101-1
[11] Villalba V M and Rojas C 2006 *Int. J. Mod. Phys.* A **21** 313–25
[12] Villalba V M and Rojas C 2007 *Phys. Lett.* A **362** 21–5
[13] Guo J Y, Yu Y and Jin S W 2009 *Cent. Eur. J. Phys.* **7** 168–74
[14] Jiang Y, Dong S, Antillon A and Cassou M L 2006 *Eur. Phys. J.* C **45** 525–28
[15] Sogut K, Havare A and Acikgoz I 2002 *J. Math. Phys.* **43** 3952
[16] Kozack R E, Clark B C, Hama S, Mishra V K, Mercer R L and Ray L 1989 *Phys. Rev.* C **40** L2181
[17] Yasuk F, Berkdemir C, Berkdemir A and Onem C 2005 *Phys. Scr.* **71** 340–3
[18] Merad M 2007 *Int. J. Theor. Phys.* **46** 2105
[19] Boumali A 2008 *Can. J. Phys.* **86** 1233

[20] Hulthen L 1942 *Ark. Mat. Astron. Fys.* **28A** 5
[21] http://eom.springer.de/h/h110350.htm
[22] Petiau G 1936 *Acad. R. Belg. Cl. Sci. Mem. Collect.* **8** 16
Kemmer N 1938 *Proc. R. Soc.* A **166** 127
Kemmer N 1939 *Proc. R. Soc.* A **173** 91
Duffin R J 1939 *Phys. Rev.* **54** 1114
[23] Cheraitia B B and Boudjedaa T 2005 *Phys. Lett.* A **338** 97–107
[24] Barut A O 1990 *Phys. Lett.* B **237** 436–9
[25] Unal N 1997 *Found. Phys.* **27** 747–58
Unal N 1998 *Found. Phys.* **28** 755–62
[26] Unal N 2005 *Concepts Phys.* **2** 273–82
[27] Abramowitz M and Stegun I A 1970 *Handbook of Mathematical Functions* (New York: Dover)